Duality

Reminder

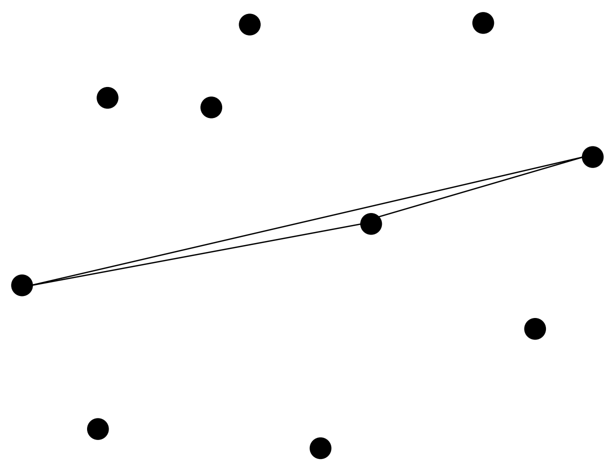
- Point line duality – Demo
- $(a, b) \sim y = ax - b$
- Given a point p and a line ℓ the duals are D_p, D_ℓ
- If p is above ℓ then D_p is **below** D_ℓ
- If three points are collinear the dual lines intersect in a single point

Warm up

- We have seen that the dual of a line segment is a left-right double wedge
- What type of object in the primal plane would dualize to a top-bottom double wedge?
  - The complement of a line segment
- What is the dual of the collection of points inside a given triangle with vertices p, q, and r?
  - The union of the three double wedges of the triangle edges

Minimal area triangle

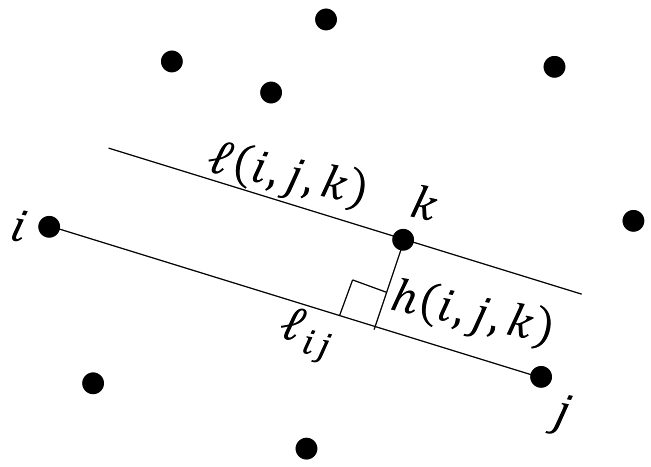
- Given a set of points, we want to find the smallest triangle (by area)



- Naïve algorithm - $O(n^3)$, can we do better?

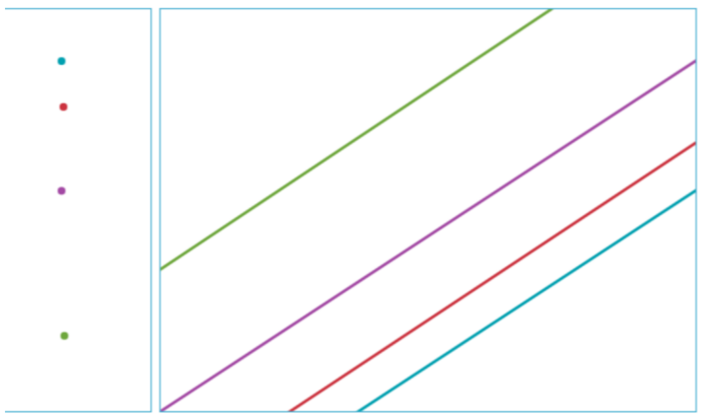
Minimal area triangle

- Notations:
- ℓ_{ij} - the segment between p_i and p_j
- $h(i,j,k)$ - the distance between ℓ_{ij} and p_k
- $\ell(i,j,k)$ - the line passing through p_k and parallel to ℓ_{ij}



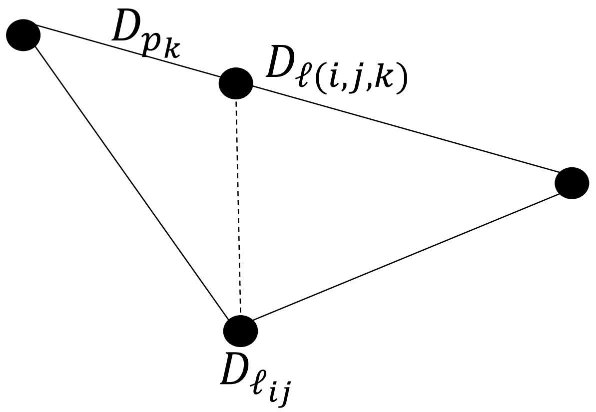
Minimal area triangle

- Notice that $h(i, j, k)$ is equal to the distance between ℓ_{ij} and $\ell(i, j, k)$
- What are parallel lines are mapped to in the dual plane?
  - Points with the same x coordinate
- For a given i, j, we only need to consider p_k which creates lines $\ell(i, j, k)$ directly above or below i, j
- This problem is easier in the dual plane.



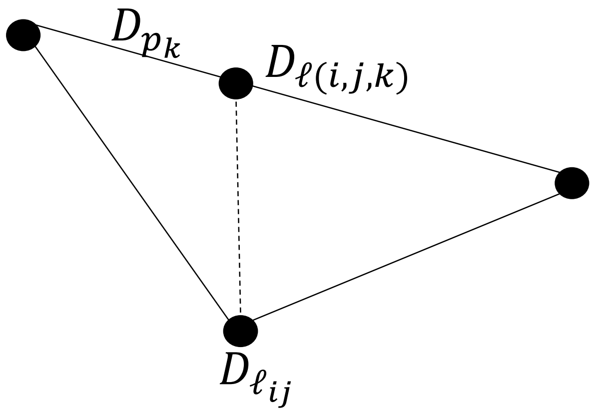
Minimal area triangle

- The dual of ℓ_{ij} is the intersection of D_{p_i} and D_{p_j}
- The dual of $\ell(i,j,k)$ is the point of D_{p_k} with the same x coordinate as $D_{\ell_{ij}}$
- Thus, for each ℓ_{ij} we need to consider the points on the segments directly above and below $D_{\ell_{ij}}$ in the dual plane



Minimal area triangle

- A better algorithm will be as follow:
- Sweep all the lines in the dual plane (i.e. all D_{p_i})
- For each intersection ($O(n^2)$) , look for the lines directly below and above ($O(\log n)$) and calculate the area of the relevant triangles
- Total time: $O(n^2 \log n)$
- Can be improved to $O(n^2)$ using topological line sweeping (Edelsbrunner and Guibas, 1989)



Sylvester-Gallai theorem

- In 1893 the following question was raised by Sylvester in a column of mathematical problems:

QUESTIONS FOR SOLUTION.

11851. (Professor SYLVESTER.)—Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.

- In 1943 Erdős raised the problem again. It was proven by Gallai in 1944
- However in 1941, Melchior proved the following theorem:
  Given n lines in the plane, there exist at least three intersection points determined by exactly two lines
- Melchior’s theorem is the dual version of Sylvester-Gallai theorem (actually, it is slightly stronger)

Sylvester-Gallai theorem

- Melchior's proof for the dual version:
- Consider the planar graph created by a set of n lines
  - That is, an edge is a line **segment** between two intersection points
- Notice that each face have at least 3 edges, and each edge bounds 2 faces, thus: $2E \geq 3F \Rightarrow F \leq \frac{2E}{3}$
- Plugging thus into Euler's characteristic we get $E \leq 3V - 3$
- However, if each vertex was the intersection of 3 lines, there will be at least $3V$ edges, which is a contradiction.

Sylvester-Gallai theorem

- In 1958 Kelly published another proof for the primal version which is considered to be the most elegant proof:
- Consider the point-line pair (P_0, ℓ_0) which minimizes the distance between the point and line.
- Claim: the line ℓ_0 is determined by only two points.
- Otherwise there will two points on ℓ_0 on one side of Q (see picture)
- The distance between P_1 and the line between P_0 and P_2 is less than the distance between P_0 and ℓ_0